

A NECESSARY AND SUFFICIENT CONDITION FOR THE ASYMPTOTIC VERSION OF AHLFORS' DISTORTION PROPERTY¹

BY

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ABSTRACT. Let f be a conformal map of $R = \{w = u + iv \in \mathbb{C} \mid \varphi_0(u) < v < \varphi_1(u)\}$ onto $S = \{z = x + iy \in \mathbb{C} \mid 0 < y < 1\}$ where the $\varphi_j \in C^0(-\infty, \infty)$ and $\operatorname{Re} f(w) \rightarrow \pm \infty$ as $\operatorname{Re} w \rightarrow \pm \infty$. There are well-known results giving conditions on R sufficient for the distortion property $\operatorname{Re} f(u + iv) = \int_0^u (\varphi_1 - \varphi_0)^{-1} du + \operatorname{const.} + o(1)$, where $o(1) \rightarrow 0$ as $u \rightarrow +\infty$. In this paper the authors give a condition on R which is both necessary and sufficient for f to have this property.

Let $R \subset \mathbb{C}$ be a region of the form $R = \{w = u + iv \mid \varphi_0(u) < v < \varphi_1(u)\}$ where φ_0 and φ_1 are extended real valued functions defined for $-\infty < u < +\infty$. For the sake of simplicity we shall require φ_0 and φ_1 to be continuous. Let $\theta(u) = \varphi_1(u) - \varphi_0(u)$.

Let $S \subset \mathbb{C}$ be the parallel strip $S = \{z = x + iy \mid 0 < y < 1\}$. Let $w = F(z)$ be a one-to-one conformal map of S onto R such that $\operatorname{Re} F(z) \rightarrow \pm \infty$ as $\operatorname{Re} z \rightarrow \pm \infty$, respectively. Let $z = f(w)$ be the inverse mapping.

Inequalities of the form

$$(a) \quad m \leq \operatorname{Re} f(w) - \int_0^u \frac{du}{\theta(u)} \leq M \quad (w = u + iv)$$

were first investigated in Ahlfors [1] (the left hand inequality corresponds to his *Distortion Theorem*; the right hand inequality to his *Second Fundamental Inequality*). That work stimulated efforts to find other properties of R which imply the validity of (a).

The problem takes a simpler form when Eke's theorem [2, Theorem 2] is applied (cf. also [8, Theorem 3]). This theorem shows that (a) holds for $0 < u < \infty$ if and only if the center term actually tends to a limit

$$(A) \quad \operatorname{Re} f(w) = \int_0^u \frac{du}{\theta(u)} + \operatorname{const.} + o(1) \quad (w = u + iv),$$

where $o(1) \rightarrow 0$ as $\operatorname{Re} w \rightarrow +\infty$.

Ahlfors' original results, as strengthened by Jenkins-Oikawa [4], show that (A) holds if R contains the real axis and its boundary curves φ_0, φ_1 are of bounded

Received by the editors December 1, 1981 and, in revised form, March 2, 1982.

1980 *Mathematics Subject Classification*. Primary 30A30.

¹Research supported in part by the National Science Foundation.

variation and bounded away from zero. A number of other geometric properties of R which imply (A) have been discovered. Examples of such sufficient conditions are the bounded $2/3$ -variation condition of Jenkins-Oikawa [5], the convergence of the integral $\int_0^\infty \varphi_j'^2 (1 + |\varphi_j'|)^{-1} du$ for $j = 0, 1$ (Lelong-Ferrand [3], Rodin-Warschawski [10]), and the convergence of $\int_0^\infty (\varphi_0'^2 + \varphi_1'^2) \theta^{-1} du$ (Warschawski [11], Rodin-Warschawski [8]). None of these sufficient conditions is also necessary.

In Theorem 1 below we give a complete solution to the problem of finding geometric conditions on R which are both necessary and sufficient for property (A).

REMARK. The sufficient condition of [11] referred to above can be derived directly from Theorem 1 by taking $\alpha(u, t) = t\varphi_1(u) + (1 - t)\varphi_0(u)$. It is not evident if the other sufficient conditions can be derived from Theorem 1 in a direct manner.

1. A class C^1 real valued function $\alpha(u, t)$ defined for $-\infty < u < +\infty$, $0 < t < 1$ will be called a *stratification* of R if $u + i\alpha(u, t) \in R$ for all (u, t) in the domain of α . For simplicity we shall also require $\alpha_t > 0$.

THEOREM 1. *A necessary and sufficient condition for R to have property (A) is that R admit a stratification $\alpha(u, t)$ such that the integrals*

$$(1) \quad \int_0^1 \int_{u'}^{u''} \left(\frac{1}{\theta(u)} - \frac{1}{\alpha_t(u, t)} \right) du dt$$

and

$$(2) \quad \int_0^1 \int_{u'}^{u''} \frac{\alpha_u^2(u, t)}{\alpha_t(u, t)} du dt$$

remain bounded above and below as $u', u'' \rightarrow +\infty$.

PROOF. *Necessity.* Since the angle of inclination of any chord of a boundary component of R is less than $\pi/2$ in magnitude, it follows that $|\operatorname{Arg} F'(z)| < \pi/2$; a detailed proof is given in Lemma 2 of §2. Hence for each fixed $t \in (0, 1)$ the stream line $\{w \in R \mid \operatorname{Im} f(w) = t\}$ is the graph of a function. Denote this function by $u \mapsto \alpha(u, t)$. We shall show that this α is a stratification of R which satisfies the boundedness conditions of the theorem.

By the definition of α we have

$$(3) \quad v(x, t) = \alpha(u(x, t), t)$$

where $F(x + it) = u(x, t) + iv(x, t)$. Take partial derivatives with respect to x, t in (3) to obtain

$$(4) \quad \alpha_u(u, t) = -\frac{\operatorname{Im} f'(w)}{\operatorname{Re} f'(w)}, \quad \alpha_t(u, t) = \frac{1}{\operatorname{Re} f'(w)} \quad (w = u + i\alpha(u, t)).$$

The integral (2) for this α can be estimated as follows, where $u' < u''$:

$$\begin{aligned} \int_0^1 \int_{u'}^{u''} \frac{\alpha_u^2}{\alpha_t} du dt &= \int_0^1 \int_{u'}^{u''} \frac{\operatorname{Im}^2 f'}{\operatorname{Re} f'} du dt \\ &= \int_0^1 \int_{x(u', \alpha(u', t))}^{x(u'', \alpha(u'', t))} \frac{\operatorname{Im}^2 f'}{\operatorname{Re} f'} (\cos \operatorname{Arg} F') |F'| dx dt \\ &= \int_0^1 \int_{x(u', \alpha(u', t))}^{x(u'', \alpha(u'', t))} \sin^2 \operatorname{Arg} F' dx dt \\ &\leq \int_0^1 \int_{x_0}^{+\infty} \operatorname{Arg}^2 F' dx dt \end{aligned}$$

for suitable x_0 . By Theorem 5 of Rodin-Warschawski [9] (see Lemma 1 in §2 for a self-contained proof of this fact in the present, less general, context) the last integral is finite under our hypothesis that R satisfies property (A).

The integral (1) for this α can be transformed as follows:

$$\begin{aligned} \int_0^1 \int_{u'}^{u''} \left(\frac{1}{\theta(u)} - \frac{1}{\alpha_t(u, t)} \right) du dt &= \int_0^1 \int_{u'}^{u''} \left(\frac{1}{\theta(u)} - \operatorname{Re} f' \right) du dt \\ &= \int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x(u', \alpha(u', t))}^{x(u'', \alpha(u'', t))} \operatorname{Re} f' \cdot \cos \operatorname{Arg} f' \cdot |F'| dx dt \\ &= \int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x'}^{x''} \cos^2 \operatorname{Arg} F' dx dt \\ &= \int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x'}^{x''} dx dy + \int_0^1 \int_{x'}^{x''} \sin^2 \operatorname{Arg} F' dx dt \end{aligned}$$

where $x' = x(u', \alpha(u', t))$ and $x'' = x(u'', \alpha(u'', t))$. As already noted, this last integral is bounded under the assumption that property (A) holds. The remaining term

$$\int_{u'}^{u''} \frac{du}{\theta(u)} - \int_0^1 \int_{x'}^{x''} dx dy = \int_{u'}^{u''} \frac{du}{\theta(u)} - \iint_{f(R(u', u''))} dx dy,$$

where $R(u', u'') = \{w \in R \mid u' < \operatorname{Re} w < u''\}$, is also bounded. Indeed, note that property (A) implies that the horizontal oscillation

$$\omega(u) = \sup\{\operatorname{Re} f(w_2) - \operatorname{Re} f(w_1) \mid w_1, w_2 \in R \text{ and } \operatorname{Re} w_1 = \operatorname{Re} w_2 = u\}$$

tends to zero as $u \rightarrow +\infty$. Let $x'' - x' = \operatorname{Re} f(u'' + iv'') - \operatorname{Re} f(u' + iv')$ where $u' + iv' \in R$, $u'' + iv'' \in R$, and $u' < u''$. Then the assertion of boundedness follows from

$$x'' - x' - \omega(u') - \omega(u'') \leq \iint_{f(R(u', u''))} dx dy \leq x'' - x' + \omega(u') + \omega(u''),$$

the consequence of property (A),

$$x'' - x' = \int_{u'}^{u''} \frac{du}{\theta(u)} + O(1),$$

and the fact $\omega(u) = o(1)$ mentioned above.

Sufficiency. We now assume that $\alpha(u, t)$ is a stratification of R such that the integrals (1) and (2) are bounded. For given $0 \leq u' < u''$ consider the curve family $\{\gamma_t\}_{0 < t < 1}$ defined by $u \mapsto \gamma_t(u) = u + i\alpha(u, t)$ for $u' \leq u \leq u''$. By well-known properties of extremal length we have

$$\int_{u'}^{u''} \frac{du}{\theta(u)} \leq \lambda_R(u', u'') \leq \lambda(\{\gamma_t\}_{0 < t < 1}),$$

where $\lambda_R(u', u'')$ is the extremal distance between the vertical sides in $\{w \in R \mid u' < \operatorname{Re} w < u''\}$. We shall show that $\lambda(\{\gamma_t\}_{0 < t < 1}) \leq \int_{u'}^{u''} \theta^{-1}(u) du + O(1)$ where $O(1)$ is bounded for all $0 \leq u' < u''$. It will then follow that property (A) holds (see Theorem 3 of Rodin-Warschawski [8]; cf. also Eke [2, Theorem 2]).

Since $\{\gamma_t\}_{0 < t < 1}$ is a 1-parameter curve family one can calculate its extremal length exactly (see, for example, Theorem 14 of [6]). Define a map of $\{0 < u < +\infty, 0 < t < 1\}$ into R , denoted $u + it \mapsto c(u, t) = u + iv$, by letting $v = \alpha(u, t)$. Then

$$\begin{aligned} J(u, t) &\equiv \frac{\partial(u, v)}{\partial(u, t)} = \begin{vmatrix} 1 & 0 \\ \alpha_u & \alpha_t \end{vmatrix} = \alpha_t, \\ \left| \frac{\partial c(u, t)}{\partial u} \right|^2 &= 1 + \alpha_u^2, \\ l(t) &\equiv \int_{u'}^{u''} \frac{|\partial c / \partial u|^2}{J} du = \int_{u'}^{u''} \frac{1 + \alpha_u^2}{\alpha_t} du. \end{aligned}$$

One has

$$\begin{aligned} \lambda(\{\gamma_t\}) &= \left(\int_0^1 \frac{dt}{l(t)} \right)^{-1} \leq \int_0^1 l(t) dt = \int_0^1 \int_{u'}^{u''} \left(\frac{1 + \alpha_u^2}{\alpha_t} \right) du dt \\ &= \int_0^1 \int_{u'}^{u''} \frac{1}{\alpha_t} du dt + \int_0^1 \int_{u'}^{u''} \frac{\alpha_u^2}{\alpha_t} du dt. \end{aligned}$$

Our hypothesis on the boundedness of integrals (1) and (2) means that the sum of these last two integrals is equal to $\int_{u'}^{u''} \theta^{-1} du$ plus bounded terms. Hence $\lambda(\{\gamma_t\}) \leq \int_{u'}^{u''} \theta^{-1} du + O(1)$ as desired.

2. We now prove the two lemmas referred to in the necessity part of the proof of Theorem 1.

LEMMA 1. *Suppose $f: R \rightarrow S$ satisfies property (A). Then the inverse function $F: S \rightarrow R$ satisfies*

$$\iint_{\substack{0 < x < \infty \\ 0 < y < 1}} \operatorname{Arg}^2 F'(z) dx dy < \infty.$$

PROOF. Let $R(a, b) = R \cap \{w \mid a < \operatorname{Re} w < b\}$. Let $l(u)$ be the length of $f(\theta_u)$, where $\theta_u = \{w \mid \operatorname{Re} w = u, \varphi_0(u) < \operatorname{Im} w < \varphi_1(u)\}$. We have

$$\begin{aligned} 0 &\leq \iint_{R(0, u)} \left[\frac{1}{\theta(u)} - |f'(w)| \right]^2 du dv \\ &= \int_0^u \frac{du}{\theta(u)} - 2 \int_0^u \frac{l(u)}{\theta(u)} du + \iint_{R(0, u)} |f'(w)|^2 du dv \\ &= \left\{ \iint_{R(0, u)} |f'(w)|^2 du dv - \int_0^u \frac{du}{\theta(u)} \right\} - 2 \int_0^u \frac{l(u) - 1}{\theta(u)} du. \end{aligned}$$

We have already seen that the term in braces is uniformly bounded for $0 < u < \infty$ (see the last paragraph of the *Necessity* part of the proof of Theorem 1). The last integral above is nonnegative since $l(u) \geq 1$. Hence

$$(5) \quad \iint_{R(0, u)} \left[\frac{1}{\theta(u)} - |f'(w)| \right]^2 du dv = O(1),$$

$$(6) \quad \int_0^u \frac{l(u) - 1}{\theta(u)} du = O(1).$$

(REMARK. With more work one can show that $\int_0^u (l^2(u) - 1)/\theta(u) du = O(1)$; see Theorem 1 of [9].)

For $0 < t < 1$ let γ_t be the part of the stream line $\{w \mid \operatorname{Im} f(w) = t\}$ which lies in $R(0, u)$. γ_t is a connected set since $|\operatorname{Arg} F'(z)| < \pi/2$. We have

$$\int_0^u \frac{du}{\theta(u)} \leq \int_{\gamma_t} \frac{|dw|}{\theta(u)} = \int_{f(\gamma_t)} \frac{|F'(z)| dx}{\theta(u(z))}.$$

After integrating for $t \in (0, 1)$ we obtain

$$\int_0^u \frac{du}{\theta} \leq \int_0^1 \int_{\gamma_t} \frac{|dw|}{\theta} dt = \iint_{f(R(0, u))} \frac{|F'|}{\theta} dx dy = \iint_{R(0, u)} \frac{|f'|}{\theta} du dv.$$

The last integral can be rewritten as $\int_0^u l(u) \theta^{-1}(u) du$ which, in view of (6), is equal to $\int_0^u \theta^{-1}(u) du + O(1)$. We conclude that

$$\int_0^1 \int_{\gamma_t} \frac{|dw| - du}{\theta(u)} dt = O(1).$$

Replace du by $|dw| \cos \operatorname{Arg} f'(w)$ and transform the above integral to

$$\int_0^1 \int_{\gamma_t} \frac{1 - \cos \operatorname{Arg} f'(w)}{\theta} |dw| dt = \iint_{f(R(0, u))} (1 - \cos \operatorname{Arg} f'(w(z))) \frac{|F'|}{\theta} dx dy.$$

Thus

$$(7) \quad \iint_{f(R(0, u))} (1 - \cos \operatorname{Arg} F'(z)) \frac{|F'|}{\theta} dx dy = O(1).$$

A change of variables in (5) leads to

$$(8) \quad \iint_{f(R(0, u))} \left(\frac{F'}{\theta} - 1 \right)^2 dx dy = O(1).$$

It follows from (7) and (8) that

$$(9) \quad \iint_{f(R(0, u))} (1 - \cos \operatorname{Arg} F'(w)) dx dy = O(1);$$

indeed, (8) shows that the set $E_1 = \{z \mid |F'(z)|/\theta(u(z)) \leq \frac{1}{2}\}$ has finite area and hence

$$\iint_{E_1} (1 - \cos \operatorname{Arg} F'(z)) dx dy < \infty.$$

On the complementary set $E_2 = \{z \mid |F'(z)|/\theta(u(z)) > \frac{1}{2}\}$ equation (7) shows that

$$\iint_{E_2} (1 - \cos \operatorname{Arg} F'(z)) dx dy < \infty.$$

Therefore

$$(10) \quad \iint_{\substack{0 < x < \infty \\ 0 < y < 1}} (1 - \cos \operatorname{Arg} F'(z)) dx dy < \infty.$$

The estimate $1 - \cos \beta \geq (4/\pi^2)\beta^2$ is valid in the range $|\beta| \leq \pi/2$. When this is applied to (10) we obtain $\iint \operatorname{Arg}^2 F'(z) dx dy < \infty$ as asserted. This completes the proof of Lemma 1.

LEMMA 2. *The map $F: S \rightarrow R$ satisfies $|\operatorname{Arg} F'(z)| < \pi/2$ for all $z \in S$.*

PROOF. The proof is modeled in part after the argument in [7, pp. 102–104]. Let $R_a = R \cap \{w = u + iv \mid \operatorname{Re} u > a\}$ for some fixed a and let G map the half-strip $S_1 = \{0 < x < \infty, 0 < y < 1\}$ conformally and one-to-one onto R_a such that 0 and i correspond to $w = a + i\varphi_0(a)$ and $w = a + i\varphi_1(a)$, respectively, and

$$\lim_{x \rightarrow +\infty} \operatorname{Re} G(z) = +\infty.$$

We show first that

$$(11) \quad |\operatorname{Arg} G'(z)| \leq \frac{\pi}{2} \quad \text{for } z \in S_1.$$

For $b > a$ we consider the quadrilateral

$$Q = \{a < u < b, \varphi_0(u) < v < \varphi_1(u)\}.$$

Then there exists a unique $\beta > 0$ and a one-to-one conformal map g of the rectangle $T = \{0 < x < \beta, 0 < y < 1\}$ onto Q such that the vertices 0, β , $i\beta$ and i of T correspond to the vertices $a + i\varphi_0(a)$, $b + i\varphi_0(b)$, $b + i\varphi_1(b)$, $a + i\varphi_1(a)$, respectively. We reflect T in the line $x = \beta$ and obtain a symmetrical rectangle T' and an analytic extension of g which maps T' onto a quadrangle Q' symmetrical to Q with respect to the line $u = b$. For fixed $h > 0$ ($h < \beta$) we define now

$$(12) \quad P(z, h) = P(z, h; g) = \arg \frac{g(z+h) - g(z)}{h},$$

where the branch of the argument is determined to coincide with the principal branch at $z = 0$. The geometry of the situation shows that $|P(0, h)| < \pi/2$ and that $P(z, h)$ extends continuously to $\text{Cl } T$. As z describes the boundary of T , $|P(z, h)|$ remains bounded by $\pi/2$. Since P is harmonic in T and continuous in $\text{Cl } T$, $|P(z, h)| < \pi/2$ for all $z \in T$. Thus the continuous argument function in (12) is actually the principal branch everywhere.

We choose now a sequence $\{b_n\}$ with $b_n \nearrow +\infty$ as $n \rightarrow \infty$ and determine a corresponding sequence $\{\beta_n\}$ such that the rectangle $T_n = \{0 < x < \beta_n, 0 < y < 1\}$ is mapped conformally onto the quadrilateral $Q_n = \{a < u < b_n, \varphi_0(u) < v < \varphi_1(u)\}$ with vertices of T_n corresponding to those of Q_n as indicated above. If g_n denotes the mapping function, it follows as in [7, p. 303] that $\lim_{n \rightarrow \infty} g_n(z) = G(z)$, uniformly in any compact subset of S_1 . Hence, uniformly in any compact subset of S_1

$$P(z, h; g_n) \rightarrow P(z, h; G) = \text{Arg} \frac{G(z+h) - G(z)}{h} \quad \text{as } n \rightarrow \infty,$$

and then

$$|P(z, h; G)| \leq \pi/2 \quad \text{for } z \in S_1.$$

Letting $h \rightarrow 0$ we obtain (11).

Next we observe that f , the inverse of F , maps R_a onto a subregion $f(R_a) \subset S$ (pictured in the $\zeta = \xi + i\eta$ plane). If θ_a denotes a crosscut $\{u = a, \varphi_0(a) < v < \varphi_1(a)\}$ of R which determines R_a , then $f(R_a)$ is bounded by the arc $\gamma = f(\theta_a)$ and the two half-lines on $\eta = 0$ and $\eta = 1$ which extend from the endpoints of γ to $+\infty$. Let $\psi: S_1 \rightarrow f(R_a)$ be the one-to-one conformal map of S_1 onto $f(R_a)$ such that $z = 0$ and $z = i$ correspond to the endpoints of γ and $\lim_{x \rightarrow +\infty} \text{Re } \psi(z) = +\infty$. Then $G(z) = F(\psi(z))$ and thus $G'(z) = F'(\psi(z)) \cdot \psi'(z)$. It is an elementary fact that $\lim_{k \rightarrow \infty} \psi'(z)$ exists for unrestricted approach in S_1 and is positive. Hence given any $\varepsilon > 0$ there exists an $x_0 = x_0(\varepsilon)$ such that (by (11))

$$|\text{Arg } F'(\psi(z))| \leq |\text{Arg } G'(z)| + \varepsilon \leq \pi/2 + \varepsilon$$

for $\text{Re } z \geq x_0, 0 < y < 1$.

Returning to $f(R_a)$ in the ζ -plane we can, given ε , determine a $\xi_0 = \xi_0(\varepsilon)$ such that

$$(13) \quad |\text{Arg } F'(\zeta)| \leq \pi/2 + \varepsilon \quad \text{for } \xi \geq \xi_0(\varepsilon) \text{ and } 0 < \eta < 1.$$

In an analogous manner—by choosing R_a as the subregion of R determined by θ_a to the left of θ_a —we can establish that for every $\varepsilon > 0$ there exists a $\xi_1 = \xi_1(\varepsilon)$ such that

$$(14) \quad |\text{Arg } F'(\zeta)| \leq \pi/2 + \varepsilon \quad \text{for } \xi \leq \xi_1(\varepsilon) \text{ and } 0 < \eta < 1.$$

To complete the proof we consider the rectangle

$$(15) \quad \{\zeta = \xi + i\eta \mid \xi_1(\varepsilon) \leq \xi \leq \xi_0(\varepsilon), 0 \leq y \leq 1\}$$

for fixed $h > 0$

$$P(\zeta, h; F) = \arg \frac{F(\zeta+h) - F(\zeta)}{h}.$$

Again, we see from the geometry that for ζ on the horizontal sides of (15) we have by choosing the principal view

$$|P(\zeta, h; F)| < \pi/2.$$

We can continue $P(\zeta)$ as a harmonic function into S . Since for $\zeta \in S$

$$\arg \frac{F(\zeta + h) - F(\zeta)}{h} = \arg F'(\zeta + \alpha h), \quad 0 < \alpha < 1,$$

where the same determination of the argument is taken on both sides, we see from (13) and (14) that the continuation of P along the two vertical sides of (15) remain the principal value and that

$$|P(\zeta, h; F)| \leq \pi/2 + \varepsilon.$$

Hence we have on the boundary and therefore in the interior of the rectangle

$$\left| \operatorname{Arg} \frac{F(\zeta + h) - F(\zeta)}{h} \right| \leq \pi/2 + \varepsilon.$$

Letting $h \rightarrow 0$ we obtain $|\operatorname{Arg} F'(\zeta)| \leq \pi/2 + \varepsilon$ for ζ in (15). Since ε is arbitrary we obtain $|\operatorname{Arg} F'(s)| \leq \pi/2$ for $\zeta \in S$. By the maximum principle the strict inequality holds.

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